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Note on the Best Possible Nagumo Uniqueness Theorem for the Characteristic Boundary Value Problem for $u_{xy} = f(x, y, u)$

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1. INTRODUCTION

It is well known that for the initial value problem, $y' = f(x, y)$ with $y(x_0) = y_0$, the classical Nagumo condition,

$$|x - x_0| \, |f(x, y_2) - f(x, y_1)| \leqslant C \, |y_2 - y_1|, \tag{1}$$

implies uniqueness for $0 < C \leqslant 1$. A classical example of Perron [1] shows that this result is the best possible in the sense that uniqueness does not necessarily follow if $C > 1$ with no further assumptions on f.

Results of Diaz and Walter [2] show that the characteristic boundary value problem,

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{2a}$$

$$u(x, y_0) = \sigma(x), \tag{2b}$$

$$u(x_0, y) = \tau(y), \tag{2c}$$

$$\sigma(x_0) = \tau(y_0), \tag{2d}$$

has at most one solution if f satisfies the Nagumo condition,

$$\begin{aligned} &|x - x_0| \, |y - y_0| \, |f(x, y, u_2, p_2, q_2) - f(x, y, u_1, p_1, q_1)| \\ &\quad \leqslant \alpha(x, y) \, |u_2 - u_1| + \beta(x, y) \, |p_2 - p_1| + \gamma(x, y) \, |q_2 - q \;|, \end{aligned} \tag{3}$$

with

$$\alpha(x, y) + \beta(x, y) + \gamma(x, y) = 1. \tag{4}$$

Walter [3] has shown that in the case that α, β, and γ are all constants, the Nagumo uniqueness theorem is the best possible in the sense that if $\alpha + \beta + \gamma > 1$ then the theorem is false.

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The purpose of this note is to show explicitly that for the more restricted characteristic boundary value problem,

$$u_{xy} = f(x, y, u), \tag{5a}$$

$$u(x, y_0) = \sigma(x), \tag{5b}$$

$$u(x_0, y) = \tau(y), \tag{5c}$$

$$\sigma(x_0) = \tau(y_0) = z_0, \tag{5d}$$

there need be no assumption of continuity on f, and the Nagumo coefficient may be greater than one provided the quotient

$$\frac{f(x, y, z)}{[(x - x_0)(y - y_0)]^\beta}$$

has a limit for (x, y, z) approaching $(x_0, \bar{y}, z_0)$, $y_0 \leqslant \bar{y} \leqslant y_0 + b$, or $(\bar{x}, y_0, z_0)$, $x_0 \leqslant \bar{x} \leqslant x_0 + a$, for a fixed value of β to be specified.

2. Uniqueness Theorem

Let $a, b > 0$ be given constants and let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$. Let

$$f : [x_0, x_0 + a] \times [y_0, y_0 + b] \times \mathbf{R}$$

be real-valued and satisfy

$$|x - x_0| \, |y - y_0| \, |f(x, y, u_2) - f(x, y, u_1)| \leqslant C \, |u_2 - u_1|, \tag{6}$$

for all $x_0 \leqslant x \leqslant x_0 + a$, $y_0 \leqslant y \leqslant y_0 + b$, $-\infty < u_1, u_2 < +\infty$, with C a nonnegative constant. Let (x_0, y_0, z_0) be defined as in (5a)-(5d).

THEOREM. *If*

$$\lim \frac{f(x, y, z)}{[(x - x_0)(y - y_0)]^{\sqrt{C}-1}}$$

exists as $(x, y, z) \to (x_0, \bar{y}, z_0)$ *with* $y_0 \leqslant \bar{y} \leqslant y_0 + b$ *and as* $(x, y, z) \to (\bar{x}, y_0, z_0)$ *with* $x_0 \leqslant \bar{x} \leqslant x_0 + a$, *then there is at most one function* $u \in C\{[x_0, x_0 + a] \times [y_0, y_0 + b]\}$ *which satisfies* (5a)-(5d).

Remark. If $C = 1$ in (6) then the assumption in the theorem becomes merely that f have a limit at $(x_0, \bar{y}, z_0)$ and $(\bar{x}, y_0, z_0)$ for $x_0 \leqslant \bar{x} \leqslant x_0 + a$ and $y_0 \leqslant \bar{y} \leqslant y_0 + b$. This assumption is more familiar in the Nagumo type uniqueness theorem referenced above.

Proof of the Theorem. One may modify the proof of Diaz and Walter as follows.

Assume u and v are different solutions, and define a function

$$F : [x_0, x_0 + a] \times [y_0, y_0 + b] \to \mathbf{R}$$

by

$$F(x, y) = \frac{|u(x, y) - v(x, y)|}{[(x - x_0)(y - y_0)]^{\sqrt{C}}}, \quad \text{for } x_0 < x \leqslant x_0 + a, \quad y_0 < y \leqslant y_0 + b,$$

$$F(x, y) = 0, \quad \text{if} \quad x = x_0 \quad \text{or} \quad y = y_0 .$$

If $0 < C \leqslant 1$ then the previous Nagumo theorem implies the result.

For $C > 1$ one may prove that F is continuous as follows.

Using an extended mean value theorem of Dobrescu and Sicolovan [4], for $x_0 < x$ and $y_0 < y$,

$$\begin{aligned} F(x, y) &= \frac{|u(x, y) - v(x, y)|}{[(x - x_0)(y - y_0)]^{\sqrt{C}}} = \frac{|u_{xy}(\xi, \eta) - v_{xy}(\xi, \eta)|}{C[(\xi - x_0)(\eta - y_0)]^{\sqrt{C}-1}} \\ &= \frac{|f(\xi, \eta, u) - f(\xi, \eta, v)|}{C[(\xi - x_0)(\eta - y_0)]^{\sqrt{C}-1}} \end{aligned}$$

where $x_0 < \xi < x$ and $y_0 < \eta < y$. Consequently, using the assumption in the theorem,

$$\lim_{(x,y)\to(x_0,\bar{y})} F(x, y) = 0 = F(x_0, \bar{y}), \qquad y_0 \leqslant \bar{y} \leqslant y_0 + b,$$

and

$$\lim_{(x,y)\to(x,y_0)} F(x, y) = 0 = F(\bar{x}, y_0), \qquad x_0 \leqslant \bar{x} \leqslant x_0 + a.$$

Then, since F is obviously continuous for $x_0 < x$, $y_0 < y$, it is clear that it is also continuous on $[x_0, x_0 + a] \times [y_0, y_0 + b]$.

Pursuant to the structure of the Diaz and Walter proof, let (ξ_m, η_m) be a point where $F(x, y)$ achieves its maximum under the additional condition that $F(x, y) < F(\xi_m, \eta_m)$ whenever $x_0 + y_0 \leqslant x + y < \xi_m + \eta_m$ with $x_0 \leqslant x \leqslant x_0 + a$, $y_0 \leqslant y \leqslant y_0 + b$. Then, as before,

$$\begin{aligned} F(\xi_m, \eta_m) &= \frac{|u(\xi_m, \eta_m) - v(\xi_m, \eta_m)|}{[(\xi_m - x_0)(\eta_m - y_0)]^{\sqrt{C}}} = \frac{|u_{xy}(\xi, \eta) - v_{xy}(\xi, \eta)|}{C[(\xi - x_0)(\eta - y_0)]^{\sqrt{C}-1}} \\ &= \frac{|f(\xi, \eta, u) - f(\xi, \eta, v)|}{C[(\xi - x_0)(\eta - y_0)]^{\sqrt{C}-1}} \leqslant \frac{C\,|u(\xi, \eta) - v(\xi, \eta)|}{C[(\xi - x_0)(\eta - y_0)]^{\sqrt{C}}} = F(\xi, \eta). \end{aligned}$$

for some (ξ, η) with $x_0 < \xi < \xi_m$ and $y_0 < \eta < \eta_m$. This contradicts the fact that there is no such point $(\xi, \eta) \cdot \|$.

Added in Proof: For a more general theorem in the case of ordinary differential equations, see [5].

References

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